

Multifractal Behaviour of n -Simplex Lattice

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Abstract

We study the asymptotic behaviour of resistance scaling and fluctuation of resistance that give rise to flicker noise in an n -simplex lattice. We propose a simple method to calculate the resistance scaling and give a closed-form formula to calculate the exponent, β_L , associated with resistance scaling, for any n . Using current cumulant method we calculate the exact noise exponent for n -simplex lattices.

1 Introduction

In recent years considerable attention has been devoted to studying the properties of disordered systems with the hope of understanding percolative phenomena. Key to several such approaches has been the concept of randomness and also of frustration [1-12]. However, many of the patterns we encounter in nature are not random but self-similar and scale invariant [13-14]. For instance, the complicated and scale invariant structures that occur when a solid mixture evolves via an aggregation process [15]. To understand such systems the concept of fractals has been found to be very useful. Fractals are scale invariant objects that may be considered as intermediate lattices between regular and random (disordered) lattices [13-14,16-17]. Such a fractal lattice describes a class of random systems where the consequence of the loss of translational invariance of a lattice can be studied in detail. Additionally, resulting from their dialational symmetry, statistical, mechanical and transport problems are solvable; hence the attraction of the model in such studies [14].

In this paper we consider a particular class of fractal known as the n -simplex lattices, to model various properties of inhomogeneous materials [16-17]. The lattice is defined recursively. The map of the zero-order truncated n -simplex lattice is a complete set of $(n+1)$ points. The map of the $(r+1)$ th order n -simplex lattice is obtained by replacing each of the lattice points of the r th order map by the entire r th order map. Each of the resulting n points is connected to one of the lines connecting the original r th order vertices. The fractal and spectral dimensions of this lattice are given by:

$$\bar{d} = \frac{\ln(n)}{\ln 2} \quad (1)$$

and

$$\tilde{d} = \frac{2 \ln(n)}{\ln(n+2)}. \quad (2)$$

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The lattices with $n \geq 3$ are of particular interest as they provide a family of fractals in which \bar{d} varies with n leaving \tilde{d} almost constant.

In order to understand how resistance scales with the size of the system, in a homogeneous system, we study the distribution of currents in a network modeled by an n -simplex lattice. We consider each bond, where bond refers to a line joining two lattice points, of the zero-order network as a unit resistor offering resistance \mathbf{R} . A unit current enters the network at one of the external nodes and leaves through another, the rest of nodes being left open. It is known that the distribution of currents in such a network is found to be multifractal [18] in the sense that different moments of the distribution scale with different exponents.

For resistance scaling analysis two methods may be adopted, either (a) to obtain the distribution of current over the entire network and measure the energy dissipated in the system or (b) to simplify the network and obtain a closed-form solution. This second method has been used rigorously by us for resistance scaling and the results obtained by this method match those from the current distribution method. The moments of the current in a n -simplex are:

$$S_r^a(I_1, I_2, \dots, I_n) = \sum_p |I_p|^a, \quad (3)$$

where I_p is the current in the p th bond and p goes from 1 to $n(n-1)/2$; S_r^a is the cumulant for an arbitrary exponent a . The currents flowing in at the external nodes of a n -simplex are represented by I_1, I_2, \dots, I_n respectively (see figure 1) with the condition $I_1 + I_2 + \dots + I_n = 0$. A scaling factor independent of I_1, I_2, \dots, I_n can then be defined as:

$$\lambda(a) = \frac{S_{r+1}^a(I_1, I_2, \dots, I_n)}{S_r^a(I_1, I_2, \dots, I_n)}. \quad (4)$$

Note that $\lambda(a)$ is related to the fractal scaling exponent $D(a)$. For a fractal with a resistance scaling parameter of 2, the r th generation length scales as $L_r = 2^r$. Using the definitions $S_r^a(I_1, I_2, \dots, I_n) = L_r^{D(a)}$ and $S_r^a(I_1, I_2, \dots, I_n) \propto \lambda^r(a)$, we get:

$$D(a) = \frac{\ln \lambda(a)}{\ln 2}. \quad (5)$$

The case $a = 0$ determines the fractal dimension of the simplex because $\lambda(0)$ is simply the ratio of number of bonds in successive order of the n -simplex lattice; $a = 2$ measures the heat loss in the network and gives resistance scaling. It has been shown [19-20] that:

$$R(L) \sim L^{-\beta_l} \quad (L \gg 1) \quad (6)$$

where β_l is an exponent controlling the transport properties.

In disordered material, the elastic scattering of the carriers at impurities leads to the random conductance or resistance fluctuation. The fluctuation arises from the interference of the scattered waves, and they are random. The magnitude of the resistance noise spectrum (flicker noise 1/f) depends on a new exponent, b , pertaining to the fractal lattice. This exponent (corresponding to $a = 4$) is a member of infinite number of exponents required to characterize the fractal lattice [18]. The exact reason as to why this fluctuation occurs is unknown though it is believed that it appears in response to changes in many extrinsic parameters such as the carrier density, the applied measuring current, external electric fields and external magnetic fields. The spectrum of resistance fluctuation is given by

$$S_R(w) = \int e^{iwt} \langle R(t)R(0) \rangle dt. \quad (7)$$

The exponent b associated with the scaling behaviour of normalized noise is given by [19]:

$$\rho_R = \S_R / R^2 \sim L^{-b} \quad (L \gg 1) \quad (8)$$

As long as each bond resistance fluctuates independently with the same spectrum, the explicit frequency dependence can be discarded. The upper and lower bounds of b [19-20] are given by:

$$\beta_L < b < \bar{d}. \quad (9)$$

The paper is organised as follows: In Section 2 we derive a closed-form solution to calculate β_L for any value of n . In Section 3 we use the current cumulant method to calculate the noise exponent for n -simplex. The paper ends with a brief discussion on the bounds proposed and comparison with our results with experimental data.

2 Calculation of β_L associated with resistance scaling

In this section we propose a simple method of calculating β_l for any n -simplex. Consider a fixed current I_1 entering at one of the external nodes of the lattice, and leaving from another, all the remaining external nodes being left open ($I_2 = I_3 = \dots = I_n = 0$). We calculate the equivalent resistance between these two external nodes and establish a recursion relation between r th and $(r+1)$ th order lattices and use the Real Space Renormalization Group Technique to find the exponents [18,21-22].

From the symmetry properties of the simplex, it is apparent that all $(n-2)$ external nodes apart from those through which current enters and leaves are at equipotential. Redrawing just those bonds through which currents flow, we have $(n-1)$ parallel paths for current to flow. Of these paths, one offers unit resistance and each of the others offer twice the unit resistance (since they include two resistances in series). Hence the equivalent resistance is given by:

$$\frac{1}{R_E} = \frac{1}{R} + \left[\frac{1}{2R} + \frac{1}{2R} + \dots + (n-2) \text{ terms} \right] \quad (10)$$

where R is the unit resistance and the square bracket contains exactly $(n-2)$ identical terms. This directly leads to:

$$R_E = \frac{2R}{n}. \quad (11)$$

Now if we consider a star of n -branches, each offering a resistance of R/n , the effective resistance between any two external nodes through which current flows will be $2R/n$ as they are in series and all other nodes being left open. It is then straight forward to show using these transformation for n -simplex lattice that the following scaling holds good:

$$\lambda(2) \sim \frac{R(2L)}{R(L)} = \frac{R_{r+1}}{R_r} = \frac{n+2}{n}. \quad (12)$$

From equation (11) we know that for any n -simplex the equivalent resistance of first order is:

$$R_{E1} = \frac{2R}{n}, \quad (13)$$

combined with equation (12) gives the equivalent resistance of r th order as:

$$R_{Er} = \frac{2(n+2)^{r-1}}{n^r}. \quad (14)$$

Thus we see how, by merely knowing the simplex one can calculate the equivalent resistance of any iteration. No long winded applications of Kirchoff's Laws are required to obtain the resistance scaling. The exponent β_L is related to $\lambda(2)$ by:

$$\beta_L = \frac{\ln(1/\lambda(2))}{\ln 2}. \quad (15)$$

3 Calculation of the Flicker noise exponent on n -simplex

It has been shown that the 4^{th} moment of current distribution is associated with the noise exponent [18].

Assuming that the cumulant $S_r^4(I_1, I_2, \dots, I_n)$ can be expressed as a homogeneous polynomial of degree 4, the most general polynomial is a linear combination of P_1^4 , $P_1^2 P_2$, $P_1 P_3$, P_4 and P_2^2 . These polynomials are defined as

$$\begin{aligned} P_1 &= I_1 + I_2 + I_3 + \dots, \\ P_2 &= I_1^2 + I_2^2 + I_3^2 + \dots, \\ P_3 &= I_1^3 + I_2^3 + I_3^3 + \dots, \\ P_4 &= I_1^4 + I_2^4 + I_3^4 + \dots \end{aligned}$$

However in present case $P_1 = 0$ due to current conservation. Hence $S_r^4(I_1, I_2, \dots, I_n)$ can be written as

$$S_r^4(I_1, I_2, \dots, I_n) = A_r P_2^2(I_1, I_2, \dots, I_n) + B_r P_4(I_1, I_2, \dots, I_n) \quad (16)$$

The next step is to determine $S_{r-1}^4(I_1, I_2, \dots, I_n)$. To establish a recursion relation between $S_r^4(I_1, I_2, \dots, I_n)$ and $S_{r-1}^4(I_1, I_2, \dots, I_n)$ we obtained current distribution in an n -simplex lattice at each node. It is easy to see that the current distribution at each node is

$$\sum_{k=1}^n \left[I_k + \frac{1}{n-1} \sum_{\substack{j=1 \\ k \neq j}}^n I_k - I_j \right]$$

by current conservation. In figure 1 we have shown the current along each bond. Therefore, we can write

$$S_r^4(I_1, I_2, \dots, I_n) = S_{r-1}^4(I) + S_{r-1}^4(II) + \dots + S_{r-1}^4(n)$$

where $S_{r-1}(I)$, $S_{r-1}(II)$, are the current cumulants of $(r-1)$ th order of n -simplex lattice. I , II , represents shaded region in figure 1. Above equation can be expressed as

$$\begin{aligned} S_r^4(I_1, I_2, \dots, I_n) &= A_{r-1} P_2^2(I_1, I_2, \dots, I_n) + B_{r-1} P_4(I_1, I_2, \dots, I_n) \\ &+ A_{r-1} P_4(I_1, I_2, \dots, I_n) + B_{r-1} P_2^2(I_1, I_2, \dots, I_n) \end{aligned} \quad (17)$$

which establish the transformation relation between r and $(r-1)$ th order. Comparing equations (16) and (17) we obtain the recursion relation between the n -simplex lattices of the r and $(r-1)$ th order.

$$\begin{pmatrix} A_r \\ B_r \end{pmatrix} = \frac{1}{n^4} \begin{pmatrix} (n^3 + 2)n & n^2(n+1)^2 \\ 6 & (2n+3)n \end{pmatrix} \begin{pmatrix} A_{r-1} \\ B_{r-1} \end{pmatrix} \quad (18)$$

The eigenvalues corresponding to the transformation matrix for the n -simplex lattice are given by

$$\lambda_n^\pm(a=4) = \frac{(n^3 + 2n + 5)n \pm n(n+1)\sqrt{n^4 - 2n^3 - n^2 + 2n + 25}}{2n^4}$$

and the fractal scaling exponent corresponding to largest eigenvalue is

$$D(a=4) = \frac{\ln \lambda_n^+(a=4)}{\ln 2}.$$

4 Discussion

We have seen that various moments of branch current give rise to different exponents, namely exponent b associated with the noise amplitude, β_L associated with resistance scaling and \bar{d} associated with the mass of the fractal. The relation between $D(4)$ obtained in Section 3 and the exponent b for normalised noise is as follows:

$$\frac{S_R}{R_r^2} \sim \rho_R \sim L_r^{-b}. \quad (19)$$

Now, $S_{R_r} \sim \sum_p |I_p|^4$

$$\frac{\sum_p (I_p)_r^4}{R_r^2} \sim \rho_{R_r} \sim L_r^{-b}. \quad (20)$$

This gives:

$$\frac{\rho_{R_{r+1}}}{\rho_{R_r}} = \lambda(\rho_{R_r}) \sim 2^{-b} \quad (21)$$

or,

$$b = \frac{\ln(1/\lambda(\rho_R))}{\ln 2}. \quad (22)$$

Now,

$$\lambda(\rho_R) = \frac{\sum_p (I_p)_{r+1}^4}{\sum_p (I_p)_r^4} \frac{R_r^2}{R_{r+1}^2} \quad (23)$$

or,

$$\lambda(\rho_R) = \frac{A_{r+1}}{A_r} \left(\frac{R_r}{R_{r+1}} \right)^2 \quad (24)$$

$$\lambda(\rho_R) = \frac{A_{r+1}}{A_r} \left(\frac{1}{\lambda(R)} \right)^2. \quad (25)$$

Substituting this in equation(8) gives:

$$b = \frac{\ln \left(\lambda^2(R) \times \frac{A_r}{A_{r+1}} \right)}{\ln 2}. \quad (26)$$

This expression gives the respective values of the exponent b as 1.1844, 1.0629 and 0.9269 for the 3, 4 and 5-simplex respectively. It is clear that the inequality $\bar{d} \geq b \geq -\beta_L$ is satisfied in each of the three cases.

In the limit n goes to infinity $\lambda(2) = (n+2)/n$ goes to 1. This is due to the fact that a large number of parallel equi-resistance paths are available for current flow. Such a large number of paths are available that in going from one order to the next we are in effect not altering the equivalent

resistance. The exponent β_L decreases in magnitude as we go to higher dimension, implying that resistance becomes less dependent on the length of the fractal.

With regard to flicker noise, we have seen that the scaling relation becomes increasingly complex as the order of simplex is increased. The noise versus resistance exponent Q is defined by the following:

$$Q = 2 + \frac{t}{k} \quad (27)$$

where t and k are given by:

$$R \sim (\Delta p)^{-t} \quad (28)$$

and

$$\frac{S_R}{R^2} \sim (\Delta p)^{-k}. \quad (29)$$

The experimental measurements [19-20] of t and k were made on 2d-carbon-vax mixtures and found to be 2.3 ± 0.4 and 5 ± 1 respectively. The direct plot of S_R versus R leads to $S_r \sim R^Q$ where $Q = 3.7 \pm 0.2$. The value we obtain for Q is in agreement with this as is clear from Table 1.

However, similar measurements on two dimensional films and metallic films have given values of Q differing from what we predict. Perhaps instead of taking the n -simplex lattice, if one considered a 2-d Sierpinski gasket [13-14] better results could be expected. For all $n > a$, there is a finite dimension matrix whose largest eigenvalue will give the characteristic exponents. The matrix elements are function of n and hence eigenvalues will be the well defined function of n . But for $n < a$ the result will be obtained by smaller matrix. The generalization to higher value of a and rescaling factor $b > 2$ is under progress.

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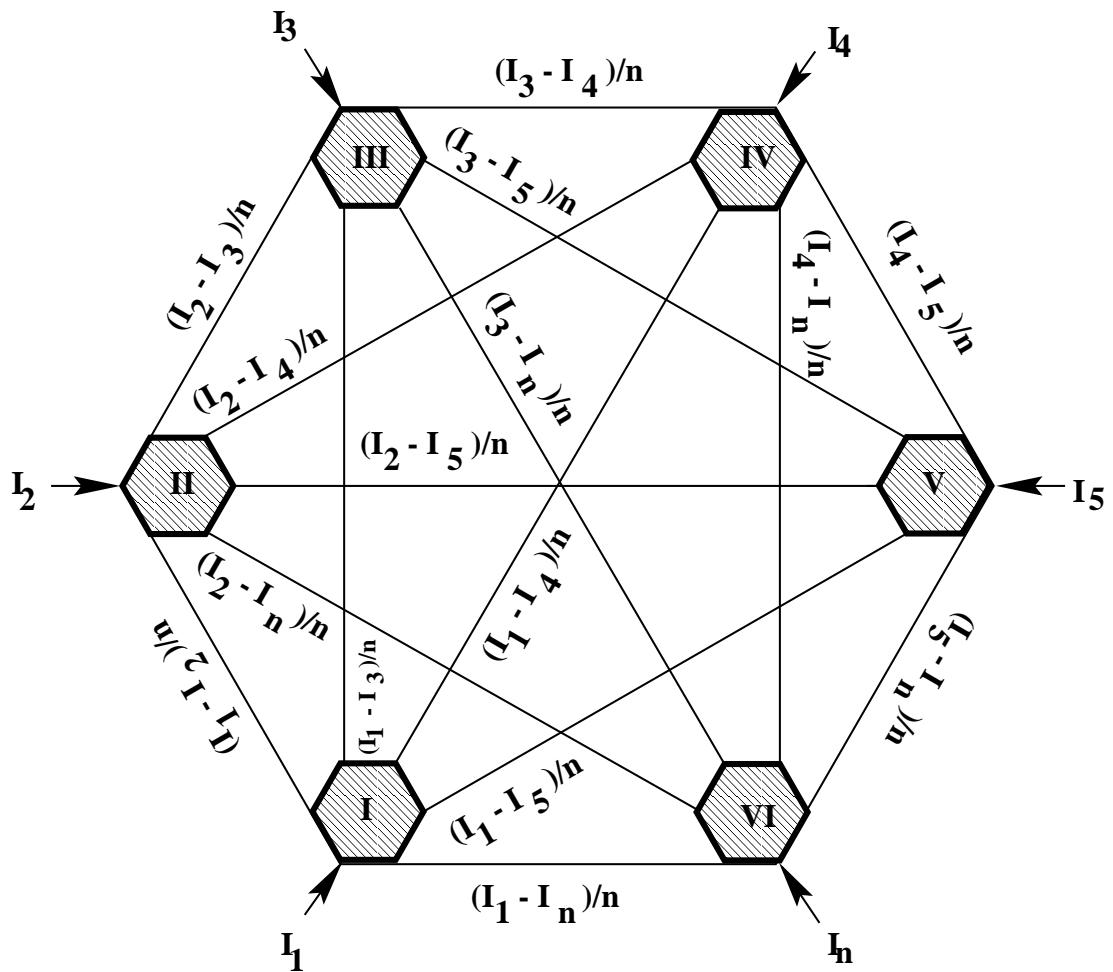


Figure 1 : Schematic representation of n -simplex lattice ($n = 6$). The current along each bond has been shown.

Table 1: *Exponents calculated for 3-, 4- and 5-simplex lattices.*

simplex	\bar{d}	b	β_L	Q
3	1.585	1.184	0.737	3.607
4	2.000	1.063	0.585	3.817
5	3.322	0.927	0.485	3.909